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The Wiener Entropy And Average Distance Of Convex Honeycomb Mesh

Xueli Su, Haizhen Ren+ and Shumin Zhang

Department of Mathematics, Qinghai Normal University, Xining 810008, Qinghai, China

Abstract. Honeycomb mesh was first put forward by Stojmenovic in 1997. Honeycomb mesh with smaller diameter and degrees of nodes (vertices), is superior to other networks, so has a great application prospect. The Wiener number is a topological index defined as the sum of distance of all pairs of vertices in the graph, which was introduced in 1947 by Harold Wiener as the path number, it is one of the most widely studied topological indices. In addition, the Wiener index is also related to a parameter of the computer network, the average distance. The convex honeycomb mesh can be depicted by a piece of an Archimedean tiling (6.6.6) that is a partial cube. Inspired by this fact, the analytical expressions for Wiener numbers of three convex honeycomb meshes and their Wiener entropies are obtained. Furthermore, their asymptotic behaviors and average distances are also discussed.

Keywords: partial cube, convex honeycomb mesh, Wiener number, entropy

1. Introduction

Honeycomb mesh was first put forward by Stojmenovic in 1997 [1]. Honeycomb mesh with smaller diameter and degrees of nodes (vertices), is superior to other networks, so has a great application prospect. Honeycomb mesh topological structure can be regarded as bipartite graph. The Wiener number (often also called the Wiener index) *W* is a topological index of *G* defined as the sum of distance of all pairs of vertices in the graph. This index was introduced in 1947 by Harold Wiener as the path number [8], it is one of the most widely studied topological indices. The Wiener index has also been widely used in organic and polymer chemistry, crystals, and drug design. A large number of literatures have been published on the calculation of Wiener indicators for various compound molecular maps. At the same time, the development of topological indicators based on distances in graphs has also been stimulated, such as the hyper-Wiener index, the Schultz index, etc. Wiener indicators in mathematics research and application can refer to [6,10]. In addition, the Wiener index is also related to a parameter of the computer network, the average distance. We note that the Wiener number and average distance are based on the node-distance topological index of the network.

For a connected graph *G*, the Wiener number of *G* is denoted by *W(G)*, and the Wiener entropy is defined by $\lim_{n\to\infty}(\log W(G)/n)$, and the average distance of G is defined by $\overline{W}(G)=\mathrm{W(G)}/C_n^2$, here n is the number of vertices of G. A plane tiling $\mathrm{T}=\{T_1,T_2,\cdots\}$ is a countable family of closed polygons which covers the plane without gaps and inner-point-overlaps, where $T_1,T_2,\cdots$ are known as tiles of T [2]. We shall restrict attention to tilings that are edge-to-edge [3, 4], it means that the mutual relation of any two tiles in T must be just one of the following three:

1. they are disjoint (have no point in common);
2. they have precisely one common point which is a vertex of each of the polygons;
3. they share a segment that is an edge of each of the two polygons.

+ Corresponding author. Tel.: +09716307622; fax: +09716307622.
E-mail address: haizhenr@126.com.

Hence a point of the plane that is a vertex of one of the polygons in an edge-to-edge tiling is also a vertex of every other polygon to which it belongs, such a point is called a vertex of the tiling T. Similarly, each edge of one of the polygons is an edge of precisely one other polygon and we call it an edge of the tiling T. Conveniently, the edges and vertices of an edge-to-edge tiling can be seen as in a graph.

Besides, we denote the type of a vertex by $a,b,c,\cdots$ if it is surrounded in cyclic order by regular polygons of orders $a,b,c,\cdots$. If the tilings with only a single type of vertex $a,b,c,\cdots$, we shall denote the tiling by $(a,b,c,\cdots)$.

Our present paper is stimulated by [3] and [4], in their studies the convex honeycomb mesh is depicted by a piece of (6.6.6) tiling(see Fig. 1(1)). We first compute the Wiener numbers of three connected subgraphs of (6.6.6)-tiling for three convex honeycomb meshes, and their exact analytical expressions are obtained, then their Wiener entropies and average distances are also considered. Our results show that if convex honeycomb meshes have different boundary conditions, then their average distances are also different, but their Wiener entropies is not affected.

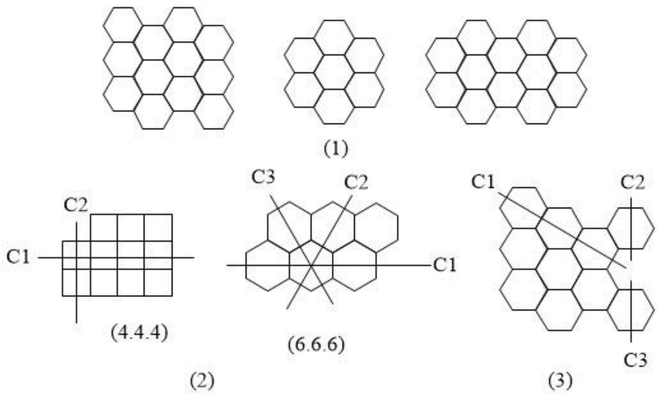

Fig. 1: (1)some connected subgraphs of tiling (6.6.6), (2) (3) elementary cut segments.

2. Preliminary

In this section, we will introduce some concepts and list some lemmas that will be used in sequel.

Definition 2.1([5]) Let G be a connected graph. The vertex set and edge set of G are denoted by V(G) and E(G) respectively, and the number of vertices of G is denoted by n(G), then the Wiener number of G is denoted as $W(G)=\sum_{u,v\subseteq V(G)} d(u,v|G)$ where *d(u,v|G)* denotes the distance between u and v, and the summation is over all pairs of vertices of G.

Definition 2.2([9]) Let G be a subgraph of one of tilings (4.4.4.4), (6.6.6), (4.8.8) and (4.6.12). Assume that G is a connected plane graph with no cut vertices, in which every interior region is bounded by a regular polygon of side length 1. An elementary cut segment *C* of *G* is a straight line segment: it is orthogonal to some edges of *G* such that touches the perimeter of *G* at the two points only, and that deleting all the parallel edges which are orthogonal to *C* and intersected by *C* will lead to exactly two connected components. (see Fig. 1(2)).

For example, there are some elementary cut segments of some connected subgraphs of (4.4.4.4) and (6.6.6) tilings in Fig.1(3). It is also easy to find that for any edge uv of G, uv must be orthogonal to an elementary cut segment C. We use C_{uv} to denote the family of edges which are orthogonal to C and intersected by C. We call C_{uv} the elementary cut associated with uv.

Lemma 2.1 ([8]) Only four tilings in Archimedean tilings, which are (4.4.4.4), (6.6.6), (4.8.8)and (4.6.12) tilings, all their connected subgraphs are partial cubes.

Definition 2.3([9]) For a partial cube G, let $C_1,C_2,\cdots,C_k$ be the elementary cut segments of G, and $G_{C_i}^0$, $G_{C_i}^1$ are two connected components of $G\setminus C_i(i=1,2,\cdots,k)$, define a label for G, $l:V(G)\to\{0,1\}^k$, for $u\in V(G)$, then the i th position of $l(u)=(l_1(u),l_2(u),\cdots,l_k(u))$ is:

$$l_i(u)=\begin{cases}0, & if\ u\in G_{C_i}^0\\ 1, & if\ u\in G_{C_i}^1\end{cases}$$

Lemma 2.2 ([7]) Let G be a partial cube on n vertices and $C_1, C_2, \cdots, C_k$ be the elementary cut segments of G. Then $W(G)=\sum_{C_i} n(G^0_{C_i})n(G^1_{C_i})$, where $n(G^0_{C_i})$ and $n(G^1_{C_i})$ are the number of vertices in the two connected components of $G \setminus C_i (i=1,2,\cdots,k)$, and the summation is over all elementary cut segments of G.

3. The Wiener number of convex honeycomb meshes

3.1. The first class of convex honeycomb meshes

See Fig.2(1), there are some connected subgraphs of (6.6.6) tiling. We can use the method mentioned in Lemma 2.2 to compute their Wiener numbers. There are three groups of elementary cut segments. One is the vertical cut segments labelled by $C_i (i=1,2,\cdots,4n-3)$ from left to right in Fig.2(1). Then we can obtain other two groups of oblique elementary cut segments by rotating $C_i (i=1,2,\cdots,4n-3)$ by $\pm 60^0$ respectively. It is easy to see that their contributions to the Wiener number of H(n) are the same as the contribution of cuts $C_i (i=1,2,\cdots,4n-3)$. Besides, the cuts C_1 and C_{4n-3}, C_2 and C_{4n-4}, $\cdots$, C_{2n-2} and C_{2n} made the same contribution to the Wiener number of *H(n)*.

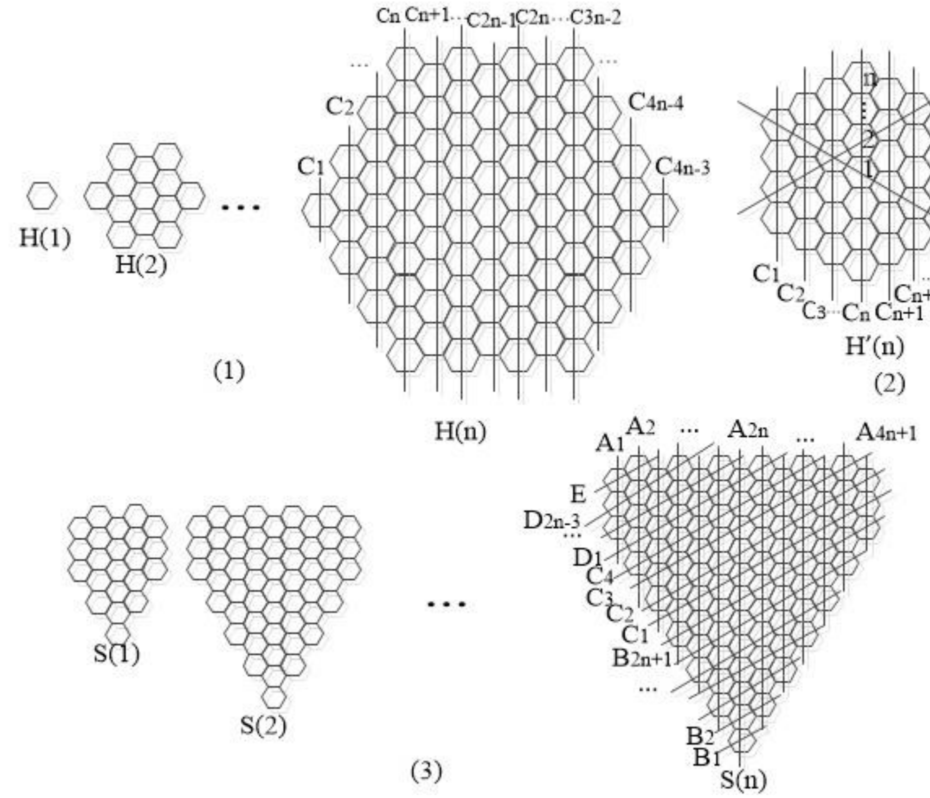

Fig. 2: three convex honeycomb meshes.

First, we calculate the number of vertices of *H(n)*.

$$
\begin{aligned}
&n(H(n+1)) - n(H(n)) = 36n,\\
&n(H(n)) - n(H(n-1)) = 36(n-1),\\
&n(H(n-1)) - n(H(n-2)) = 36(n-2),\\
&\vdots\\
&n(H(2)) - n(H(1)) = 36.
\end{aligned}
$$

Add up all the above expressions, we get $n(H(n+1)) - n(H(1)) = 36(1+2+3+\cdots+n) = 18n^2+18n$.
Therefore, $n(H(n)) = 18n^2 - 18n + 6$.

For every vertical cut $C_i (i=1,2,\cdots,n)$, we get $n(H(n)^0_{C_i}) = 2(1+4+\cdots+3i-2)+i = 3i^2$. Then, for any vertical cuts $C_i (i=n+1, n+2,\cdots,2n-2)$, we have $n(H(n)^0_{C_i}) = 3n^2 + j[2(3n-2)+1] = 3n^2+(6n-3)j, (j=i-n)$ .Finally, for elementary cut segment C_{2n-1} , we have $n(H(n)^0_{C_{2n-1}}) = n(H(n)^1_{C_{2n-1}}) = 9n^2 - 9n + 3$. Denote the contribution of $C_i (i=1,2,\cdots,4n-3)$ to the Wiener number of *H(n)* by W_1. Then we have

$$W_1 = \sum_{C_i} n(H(n)^0_{C_i}) n(H(n)^1_{C_i}) = 2\sum_{i=1}^{n} 3i^2(18n^2-18n+6-3i^2) + 2\sum_{j=1}^{n-2}[3n^2+(6n-3)j][(18n^2-18n+6)-3n^2-(6n-3)j) + (9n^2-9n+3)^2] .$$

Thus, $W_1 = 825n^5/5 - 426n^4 + 456n^3 - 258n^2 + 378n/5 - 9$. Therefore we have the Wiener number of H(n) as

$$W(H(n)) = 3W_1 = 2556n^5/5 - 1278n^4 + 1368n^3 - 774n^2 + 1134n/5 - 27 .$$

Conveniently, we can get its Wiener entropy

$$\lim_{n\to\infty} \log W(H(n))/n(H(n)) = \lim_{n\to\infty} \log(2556n^5/5 - 1278n^4 + 1368n^3 - 774n^2 + 1134n/5 - 27)/(18n^2 - 18n + 6) = 0 .$$

And its average distance $\overline{W}(H(n)) = (2556n^5/5 - 1278n^4 + 1368n^3 - 774n^2 + 1134n/5 - 27)/C^2_{18n^2-18n+6}$ .When n gets large enough, $\overline{W}(H(n))$ approximates to $142n/45$.

3.2. The second class of convex honeycomb meshes

Now, we consider another kind of connected subgraph of tiling (6.6.6), which is the second class of convex honeycomb meshes (see Fig.2(2)). We use H'(n) to denote it. The method of calculation of Wiener number of H'(n) is similar to that for H(n). For H'(n) there are three groups of elementary cut segments. As shown in Fig.2(2), one is the vertical cuts, we denote them as $C_i (i = 1,2,\cdots,2n-1)$, label it from left to right. Then we can get other two groups of oblique elementary cut segments $C_i^{'}$ and $C_i^{''}$ by rotating $C_i (i = 1,2,\cdots,2n-1)$ by $\pm 60^\circ$ respectively. It is easy to see that their contributions to the Wiener number of $H^{'}(n)$ are the same as C_i 's. Besides, C_1 and C_{2n-1} , C_2 and C_{2n-2} , $\cdots$, C_{n-1} and C_{n+1} made the same contribution to the Wiener number of $H^{'}(n)$. First, it is easy to get the number of vertices of $H^{'}(n)$ by observation, $n(H^{'}(n)) = 6[1+3+5+\cdots+(2n-1)] = 6n^2$. Then, for any vertical cuts C_i , $n(H^{'}(n)^0_{C_i}) = 2ni + i^2$.

Denote the $C_i (i = 1,2,\cdots,2n-1)$ contribution to the Wiener number of $H^{'}(n)$ by W_1 . Then we have: $W_1 = \sum_{C_i} n(H^{'}(n)^0_{C_i}) n(H^{'}(n)^1_{C_i}) = 164n^5/15 - 2n^3 + n/15$. Therefore we have the Wiener number of $H^{'}(n)$ as

$$W(H^{'}(n)) = 3W_1 = 164n^5/5 - 6n^3 + n/5 .$$

Conveniently, we can get its Wiener entropy

$$\lim_{n\to\infty} \log W(H^{'}(n))/n(H^{'}(n)) = \lim_{n\to\infty} \log(164n^5/5 - 6n^3 + n/5)/6n^2 = 0 .$$

And its average distance $\overline{W}(H^{'}(n)) = (164n^5/5 - 6n^3 + n/5)/C^2_{6n^2}$. When n gets large enough, $\overline{W}(H^{'}(n))$ approximates to $82n/45$.

3.3. The third class of convex honeycomb meshes

See Fig.2(3), where the third class of convex honeycomb meshes is illustrated. We have given two group of elementary cut segments of S(n). There are three groups of elementary cut segments. One is the vertical cut segments $A_i (i = 1,2,\cdots,4n+1)$ labelled from left to right. There are two groups of oblique elementary cut segments, one is the group of cut segments A_i and another is the group of cut segments $B_j (j = 1,2,\cdots,2n+1)$, $C_s (s = 1,2,3,4)$, $D_l (l = 1,2,\cdots,2n-3)$ and E. We can obtain another group of oblique elementary cut segments by rotating B_j , C_s , D_l and E by 60° . The number of vertices of S(n) is $n(S(n)) = 2(6n^2 + 19n + 7) = 12n^2 + 38n + 14$.

Then for every vertical cuts $A_i (i = 1,2,\cdots,2n)$, we have $n(S(n)^0_{A_i}) = 7 + 11 + 14 + \cdots + (3i+5) = -1 + 13i/2 + 3i^2/2$. Then, for vertical cuts A_{2n+1} , we have $n(S(n)^0_{A_{2n+1}}) = 6n^2 + 19n + 7$. Denote the $A_i (i = 1,2,\cdots,4n+1)$ contribution to the Wiener number of S(n) by W_1 . Then we have

$$W_1 = \sum_{A_i} n(S(n)^0_{A_i}) n(S(n)^1_{A_i}) = 49 + 6032n/15 + 1129n^2 + 3896n^3/3 + 532n^4 + 336n^5/5 .$$

Next, let us consider oblique elementary cut segments $B_j (j = 1,2,\cdots,2n+1)$. For any oblique cuts $B_j (j = 1,2,\cdots,2n+1)$, $n(S(n)^0_{B_j}) = 3 + 6 + \cdots + 3j = 3j/2 + 3j^2/2$. Denote the $B_j (j = 1,2,\cdots,2n+1)$ contribution to the Wiener number of S(n) by W_2 . Then we have $W_2 = \sum_{B_j} n(S(n)^0_{B_j}) n(S(n)^1_{B_j}) = 33 + 1067n/5 + 496n^2 + 506n^3 + 224n^4 + 168n^5/5$.

Then, we consider oblique elementary cut segments $C_s (s = 1,2,3,4)$. Denote the $C_s (s = 1,2,3,4)$ contribution to the Wiener number of S(n) by W_3 . Then it is easy to see $W_3 = \sum_{C_s} n(S(n)^0_{C_s}) n(S(n)^1_{C_s}) = -125 + 492n + 1500n^2 + 912n^3 + 144n^4$. Next, let us consider oblique elementary cut segments $D_l (l = 1,2,\cdots,2n-3)$. For any oblique cuts

$$D_l (l = 1,2,\cdots,2n-3) , \quad n(S(n)^0_{D_l}) = 3(2n+1)/2 + 3(2n+1)^2/2 + 12(2n+1) + 6 + (12n - 3l + 5)l/2 .$$

Denote the $D_l (l = 1,2,\cdots,2n-3)$ contribution to the Wiener number of S(n) by W_4 . Then we have

$$W_4 = \sum_{D_l} n(S(n)^0_{D_l}) n(S(n)^1_{D_l}) = 158 - 1859n/15 - 475n^2 + 10n^3/3 + 164n^4 + 168n^5/5 .$$

Finally, for oblique cut E, denote the cut E contribution to the Wiener number of S(n) by W_5 . Then we have $W_5 = 5(12n^2 + 38n + 9) = 45 + 190n + 60n^2$.

Therefore we have the Wiener number of S(n) as

$$W(S(n)) = W_1 + 2(W_2 + W_3 + W_4 + W_5) = 271 + 29176n/15 + 4291n^2 + 12424n^3/3 + 1596n^4 + 1008n^5/5 .$$

Conveniently, we can get its Wiener entropy

$$\lim_{n\to\infty}\log W(S(n))/n(S(n))=\lim_{n\to\infty}\log(271+29176n/15+4291n^2+12424n^3/3+1596n^4+1008n^5/5)/(12n^2+38n+14)=0 .$$

and its average distance $\overline{W}(S(n))=(271+29176n/15+4291n^2+12424n^3/3+1596n^4+1008n^5/5)/C^2_{12n^2+38n+14}$. When n gets large enough, $\overline{W}(S(n))$ approximates to $14n/5$.

3.4. Conclusion

From our results we can see that in general, if the boundary conditions of the convex honeycomb meshes are different, then the asymptotic behaviors of their average distances are also different. However, the asymptotic behaviors of their Wiener entropy are independent on their boundary conditions. So, we post the following question:

Which boundary condition can gearatee the different asymptotic behavior of the Wiener entropy for a type of tiling ?

4. Acknowledgements

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